

# A note on a class of exact solutions for a doubly anharmonic oscillator

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## Abstract

We examine a class of exact solutions for the eigenvalues and eigenfunctions of a doubly anharmonic oscillator defined by the potential  $V(x) = \omega^2/2x^2 + \lambda x^4/4 + \eta x^6/6$ ,  $\eta > 0$ . These solutions hold provided certain constraints on the coupling parameters  $\omega^2$ ,  $\lambda$  and  $\eta$  are satisfied.

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## 1. Introduction

The anharmonic oscillator we consider in this note has the potential  $V(x) = \frac{1}{2}\omega^2x^2 + \frac{1}{4}\lambda x^4 + \frac{1}{6}\eta x^6$ , where  $\omega^2$ ,  $\lambda$  and  $\eta > 0$  are real parameters. With  $E$  denoting the energy eigenvalue, the associated Schrödinger equation is

$$\frac{d^2\psi(x)}{dx^2} + (2E - \omega^2x^2 - \frac{1}{2}\lambda x^4 - \frac{1}{3}\eta x^6) \psi(x) = 0 \quad (-\infty < x < \infty) \quad (1.1)$$

subject to the boundary conditions

$$\lim_{x \rightarrow \pm\infty} \psi(x) = 0.$$

In [1], Flessas obtained exact solutions for the energy eigenvalues and eigenfunctions (for the ground state and first excited state) when certain constraints on the coupling parameters are satisfied. Singh *et al.* [4] subsequently showed how an infinite set of such solutions could be constructed, although they presented details only of the eigenfunctions with 2 and 3 nodes. Again, these solutions are obtained when a sequence of constraints on the coupling parameters is satisfied; a finite number of such solutions is engendered by the fulfillment of each set of constraints.

In this note we consider the case of eigensolutions with up to  $2N$  and  $2N + 1$  nodes, where  $N \leq 3$ , in more detail.

## 2. Solution scheme

We define the quantities

$$a = \frac{\lambda}{4} \left( \frac{3}{\eta} \right)^{\frac{1}{2}}, \quad b = \left( \frac{\eta}{3} \right)^{\frac{1}{2}}, \quad c = \omega^2 + (3\eta)^{\frac{1}{2}} - \frac{3\lambda^2}{16\eta}. \quad (2.1)$$

With the substitution

$$\psi(x) = \exp \left[ -\frac{1}{2}ax^2 - \frac{1}{4}bx^4 \right] y(x) \quad (2.2)$$

in (1.1), we obtain the differential equation

$$y''(x) - 2(ax + bx^3)y'(x) + (2E - a - cx^2)y(x) = 0. \quad (2.3)$$

We look for polynomial solutions of (2.3) of the form

$$y(x) = \sum_{n=0}^N A_n x^{2n+\epsilon} \quad (A_0 = 1; N = 0, 1, 2, \dots), \quad (2.4)$$

where  $\epsilon = 0$  (even solutions) or  $\epsilon = 1$  (odd solutions) and the  $A_n$  are coefficients to be determined. Substitution of (2.4) into the differential equation (2.3) then yields

$$\sum_{n=0}^N A_n \left\{ (2n+\epsilon)(2n-1+\epsilon)x^{2n-2} + (2E - a - 2a(2n+\epsilon))x^{2n} - (c + 2b(2n+\epsilon))x^{2n+2} \right\} = 0.$$

The requirement that the constant terms and the coefficient of  $x^{2N+2}$  should vanish produces

$$\begin{aligned} (1+\epsilon)(2+\epsilon)A_1 + 2E - a(1+2\epsilon) &= 0, \\ c + 2b(2N+\epsilon) &= 0. \end{aligned} \quad (2.5)$$

Thus we have the conditions

$$E = \frac{1}{2}a(1+2\epsilon) - \frac{1}{2}(1+\epsilon)(2+\epsilon)A_1 \quad (2.6)$$

and

$$\gamma := \left( \frac{3}{\eta} \right)^{\frac{1}{2}} \left( \frac{3\lambda^2}{16\eta} - \omega^2 \right) = 4N + 3 + 2\epsilon, \quad (2.7)$$

together with the  $N$  equations for the coefficients  $A_n$

$$\begin{aligned} (2n+1+\epsilon)(2n+2+\epsilon)A_{n+1} + (2E - a - 2a(2n+\epsilon))A_n \\ - (c + 2b(2n-2+\epsilon))A_{n-1} = 0 \quad (1 \leq n \leq N), \end{aligned} \quad (2.8)$$

with  $A_n = 0$  for  $n > N$ .

In what follows, we label the eigenvalues associated with the even solutions by  $E_{2m}$ ,  $m = 0, 1, 2, \dots$ , where  $E_0$  denotes the ground state eigenvalue, and those associated with the odd solutions by  $E_{2m+1}$ . A similar indexing applies to the eigenfunctions  $\psi(x)$ , which will possess either  $2m$  or  $2m+1$  nodes on the interval  $(-\infty, \infty)$ , respectively.

(i) *The case  $N = 0$ .* When  $N = 0$ , we obtain from (2.6) and (2.7) the values

$$E_0 = \frac{1}{2}a, \quad \gamma = 3 \quad \text{and} \quad E_1 = \frac{3}{2}a, \quad \gamma = 5.$$

The associated eigenfunctions are

$$\psi_0(x) = \exp\left[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4\right] \quad \text{and} \quad \psi_1(x) = x \exp\left[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4\right],$$

which possess 0 and 1 node, respectively. These are the special solutions of (1.1) obtained by Flessas [1].

(ii) *The case  $N = 1$ .* When  $N = 1$ , we obtain from (2.5), (2.6) and (2.7) for  $\epsilon = 0$  the values

$$E = \frac{1}{2}a - A_1, \quad \gamma = 7 \quad (c = -4b)$$

and from (2.8) the single equation

$$(2E - 5a)A_1 - c = 0.$$

Substitution of the above values of  $E$  and  $c$  yields the quadratic

$$A_1^2 + 2aA_1 + 2b = 0,$$

with solutions

$$A_1 = -a \pm \sqrt{a^2 + 2b}.$$

The positive root is associated with an eigenfunction with no node and so is a ground-state value [2, p. 21], viz.

$$E_0 = \frac{3}{2}a - \sqrt{a^2 + 2b}, \quad \psi_0(x) = \{1 + (\sqrt{a^2 + 2b} - a)x^2\} \exp\left[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4\right].$$

The negative root corresponds to an eigenfunction with 2 nodes and represents a case of the first even excited state:

$$E_2 = \frac{3}{2}a + \sqrt{a^2 + 2b}, \quad \psi_2(x) = \{1 - (\sqrt{a^2 + 2b} + a)x^2\} \exp\left[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4\right].$$

Both cases are associated with the value  $\gamma = 7$ , which gives a constraint between the three parameters  $\omega^2$ ,  $\lambda$  and  $\eta$ .

When  $\epsilon = 1$ , a similar procedure shows that

$$E = \frac{3}{2}a - 3A_1, \quad \gamma = 9$$

and

$$A_1 = -\frac{1}{3}a \pm \frac{1}{3}\sqrt{a^2 + 6b}.$$

Again, the positive root corresponds to the lowest odd eigenfunction (with 1 node) and the negative root to the second odd eigenfunction (with 3 nodes). Thus, we have

$$E_1 = \frac{5}{2}a - \sqrt{a^2 + 6b}, \quad \psi_1(x) = x\{1 + \frac{1}{3}(\sqrt{a^2 + 6b} - a)x^2\} \exp\left[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4\right],$$

$$E_3 = \frac{5}{2}a + \sqrt{a^2 + 6b}, \quad \psi_3(x) = x\{1 - \frac{1}{3}(\sqrt{a^2 + 6b} + a)x^2\} \exp\left[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4\right].$$

These special solutions of (1.1) were given in [4].

### 3. Exact solutions for the case $N = 2$

When  $N = 2$ , we have from (2.5), (2.6) and (2.7) for even modes ( $\epsilon = 0$ )

$$E = \frac{1}{2}a - A_1, \quad \gamma = 11 \quad (c = -8b)$$

together with, from (2.8), the two equations for the coefficients

$$12A_2 + (2E - 5a)A_1 - c = 0, \quad (2E - 9a)A_2 - (c + 4b)A_1 = 0.$$

Substitution of the above values of  $E$  and  $c$  then leads after some straightforward algebra to

$$A_1^3 + 6aA_1^2 + 8(a^2 - 2b)A_1 - 16ab = 0, \quad A_2 = \frac{2bA_1}{A_1 + 4a}. \quad (3.1)$$

The cubic equation for  $A_1$  can be written in its reduced form as

$$\chi^3 + p\chi + q = 0, \quad A_1 = \chi - 2a,$$

where  $p = -4(a^2 + 4b)$  and  $q = 16ab$ . The discriminant

$$\Delta = -4p^3 - 27q^2 = 256\{(a^2 + 4b)^3 - 27a^2b^2\} > 0,$$

so that (3.1) has three real roots. Moreover, inspection of the coefficients of (3.1) shows that there are two negative roots and one positive root. With

$$P = (-\frac{4}{3}p)^{\frac{1}{2}} = 4\left(\frac{a^2 + 4b}{3}\right)^{\frac{1}{2}}, \quad Q = \left(\frac{-27q^2}{4p^3}\right)^{\frac{1}{2}} = ab\left(\frac{a^2 + 4b}{3}\right)^{-\frac{3}{2}},$$

$$\theta = \frac{1}{3} \arcsin Q, \quad (3.2)$$

the roots are then given by

$$\chi_k = P \sin(\theta + \frac{2}{3}\pi k) \quad (k = 0, 1, 2).$$

It is easy to establish that the positive value of  $A_1$  corresponds to the root  $\chi_1$ .

Then we have the following even exact solutions: (a) a ground-state eigenvalue and eigenfunction given by

$$E_0 = \frac{5}{2}a - P \sin(\theta + \frac{2}{3}\pi), \quad \psi_0(x) = \{1 + A_1x^2 + A_2x^4\} \exp[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4], \quad (3.3)$$

where

$$A_1 = P \sin(\theta + \frac{2}{3}\pi) - 2a > 0, \quad A_2 = \frac{2b(P \sin(\theta + \frac{2}{3}\pi) - 2a)}{P \sin(\theta + \frac{2}{3}\pi) + 2a} > 0;$$

(b) a first even excited state (2 nodes) with eigenvalue and eigenfunction given by

$$E_2 = \frac{5}{2}a - P \sin \theta, \quad \psi_2(x) = \{1 - |A_1|x^2 - |A_2|x^4\} \exp[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4], \quad (3.4)$$

where

$$A_1 = P \sin \theta - 2a < 0, \quad A_2 = \frac{2b(P \sin \theta - 2a)}{P \sin \theta + 2a} < 0;$$

and (c) a second even excited state (4 nodes) with eigenvalue and eigenfunction given by

$$E_4 = \frac{5}{2}a - P \sin(\theta + \frac{4}{3}\pi), \quad \psi_4(x) = \{1 - |A_1|x^2 + A_2x^4\} \exp[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4], \quad (3.5)$$

where<sup>1</sup>

$$A_1 = P \sin(\theta + \frac{4}{3}\pi) - 2a < 0, \quad A_2 = \frac{2b(P \sin(\theta + \frac{4}{3}\pi) - 2a)}{P \sin(\theta + \frac{4}{3}\pi) + 2a} > 0.$$

A similar treatment for odd eigensolutions ( $\epsilon = 1$ ) yields

$$E = \frac{3}{2}a - 3A_1, \quad \gamma = 13 \quad (c = -10b)$$

and

$$(3A_1)^3 + 6a(3A_1)^2 + 8(a^2 - 4b)(3A_1) - 48ab = 0, \quad A_2 = \frac{2bA_1}{3A_1 + 4a}.$$

The reduced cubic is

$$\chi^3 + p\chi + q = 0, \quad 3A_1 = \chi - 2a,$$

where  $p = -4(a^2 + 8b)$  and  $q = 16ab$ . The three real roots are given by

$$\chi_k = P \sin(\theta + \frac{2}{3}\pi k) \quad (k = 0, 1, 2),$$

where now

$$P = 4\left(\frac{a^2 + 8b}{3}\right)^{\frac{1}{2}}, \quad Q = ab\left(\frac{a^2 + 8b}{3}\right)^{-\frac{3}{2}}, \quad \theta = \frac{1}{3}\arcsin Q,$$

Then we have the following odd exact solutions: (a) a first excited eigenvalue and eigenfunction (with 1 node) given by

$$E_1 = \frac{7}{2}a - P \sin(\theta + \frac{2}{3}\pi), \quad \psi_1(x) = x\{1 + A_1x^2 + A_2x^4\} \exp[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4], \quad (3.6)$$

where

$$A_1 = \frac{P}{3} \sin(\theta + \frac{2}{3}\pi) - \frac{2a}{3} > 0, \quad A_2 = \frac{2b(P \sin(\theta + \frac{2}{3}\pi) - 2a)}{3(P \sin(\theta + \frac{2}{3}\pi) + 2a)} > 0;$$

(b) a second excited state (3 nodes) with eigenvalue and eigenfunction given by

$$E_3 = \frac{7}{2}a - P \sin \theta, \quad \psi_3(x) = x\{1 - |A_1|x^2 - |A_2|x^4\} \exp[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4], \quad (3.7)$$

where

$$A_1 = \frac{P}{3} \sin \theta - \frac{2a}{3} < 0, \quad A_2 = \frac{2b(P \sin \theta - 2a)}{3(P \sin \theta + 2a)} < 0;$$

and (c) a third excited state (5 nodes) with eigenvalue and eigenfunction given by

$$E_5 = \frac{7}{2}a - P \sin(\theta + \frac{4}{3}\pi), \quad \psi_5(x) = x\{1 - |A_1|x^2 + A_2x^4\} \exp[-\frac{1}{2}ax^2 - \frac{1}{4}bx^4], \quad (3.8)$$

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<sup>1</sup>The quantity  $P \sin(\theta + \frac{4}{3}\pi) + 2a < 0$ , since  $P|\sin(\theta + \frac{4}{3}\pi)| > (4a/\sqrt{3})|\sin \frac{4}{3}\pi| = 2a$  when  $0 \leq \theta \leq \frac{\pi}{6}$ .

where

$$A_1 = \frac{P}{3} \sin\left(\theta + \frac{4}{3}\pi\right) - \frac{2a}{3} < 0, \quad A_2 = \frac{2b(P \sin(\theta + \frac{4}{3}\pi) - 2a)}{3(P \sin(\theta + \frac{4}{3}\pi) + 2a)} > 0.$$

#### 4. Exact solutions for the case $N = 3$

When  $N = 3$ , we have from (2.5), (2.6) and (2.7) for even modes ( $\epsilon = 0$ )

$$E = \frac{1}{2}a - A_1, \quad \gamma = 15 \quad (c = -12b)$$

and from (2.8) the equations for the coefficients

$$\begin{aligned} 12A_2 + (2E - 5a)A_1 - c &= 0 \\ 30A_3 + (2E - 9a)A_2 - (c + 4b)A_1 &= 0 \\ (2E - 13a)A_3 - (c + 8b)A_2 &= 0. \end{aligned}$$

This produces the quartic equation for  $w = A_1$  given by

$$w^4 + 12aw^3 + 4(11a^2 - 15b)w^2 + 24a(2a^2 - 11b)w - 36b(4a^2 - 5b) = 0, \quad (4.1)$$

$$A_2 = \frac{4bA_1(A_1 + 6a)}{(A_1 + 4a)(A_1 + 6a) - 30b}, \quad A_3 = \frac{2bA_2}{A_1 + 6a}.$$

The reduced quartic is

$$\chi^4 + p\chi^2 + q\chi + r = 0, \quad A_1 = \chi - 3a, \quad (4.2)$$

where  $p = -10(a^2 + 6b)$ ,  $q = 96ab$  and  $r = 9(a^4 + 12a^2b + 20b^2)$ . This equation possesses four real<sup>2</sup> roots. Although it is possible to express the roots in algebraic form, it was found that the resulting expressions were too complicated to be of practical use. In this case, we shall content ourselves with a numerical solution of the quartic equation (4.1).

If we choose, for example,  $\lambda = 0.50$ ,  $\eta = 0.03$  ( $a = 1.25$ ,  $b = 0.10$ ) then, from the constraint  $\gamma = 13$ , we have  $\omega^2 = 0.0625$ . The largest root of (4.1) together with the corresponding values of  $A_2$  and  $A_3$  are

$$A_1 = 0.264080, \quad A_2 = 0.021656, \quad A_3 = 0.000558.$$

We note that all the coefficients are positive and so this will result in a ground-state eigenfunction (no node) with the eigenvalue  $E_0 = \frac{5}{8} - A_1 = 0.360920$ . Values of the other solutions of (4.1), which we label  $A_n^{(2m)}$  ( $0 \leq m \leq 3$ ), and the associated even eigenvalues  $E_{2m}$  are presented in Table 1 and the corresponding eigenfunctions are

$$\psi_{2m}(x) = \left\{1 + \sum_{n=1}^3 A_n^{(2m)} x^{2n}\right\} \exp\left[-\frac{5}{8}x^2 - \frac{1}{40}x^4\right] \quad (0 \leq m \leq 3).$$

Table 1: Values of the coefficients when  $\lambda = 0.50$ ,  $\eta = 0.03$  and  $\omega^2 = 0.0625$  obtained from (4.1) and the corresponding even eigenvalues  $E_{2m}$ .

$m$	$A_1^{(2m)}$	$A_2^{(2m)}$	$A_3^{(2m)}$	$E_{2m}$
0	+0.264080	+0.021656	+0.000558	0.360920
1	-1.887128	-0.292761	-0.010432	2.512128
2	-4.899957	+1.859948	+0.143071	5.524957
3	-8.476994	+8.344491	-1.708197	9.101994

For odd modes with  $\epsilon = 1$  we have

$$E = \frac{3}{2}a - 3A_1, \quad \gamma = 17 \quad (c = -14b)$$

and the equations for the coefficients

$$\begin{aligned} 20A_2 + (2E - 7a)A_1 - (c + 2b) &= 0 \\ 42A_3 + (2E - 11a)A_2 - (c + 6b)A_1 &= 0 \\ (2E - 15a)A_3 - (c + 10b)A_2 &= 0. \end{aligned}$$

This produces the quartic equation for  $w = 3A_1$  given by

$$w^4 + 12aw^3 + (44a^2 - 100b)w^2 + 24a(2a^2 - 21b)w - 108b(4a^2 - 7b) = 0, \quad (4.3)$$

$$A_2 = \frac{4bA_1(3A_1 + 6a)}{(3A_1 + 4a)(3A_1 + 6a) - 42b}, \quad A_3 = \frac{2bA_2}{3A_1 + 6a}.$$

This equation also has four real roots and, for  $\lambda = 0.50$ ,  $\eta = 0.03$  and  $\omega^2 = 0.0625$ , the four sets of values of the coefficients  $A_n^{(2m+1)}$  ( $0 \leq m \leq 3$ ) and the corresponding odd eigenvalues  $E_{2m+1}$  are presented in Table 2. The associated eigenfunctions are

$$\psi_{2m+1}(x) = x \left\{ 1 + \sum_{n=1}^3 A_n^{(2m+1)} x^{2n} \right\} \exp \left[ -\frac{5}{8}x^2 - \frac{1}{40}x^4 \right] \quad (0 \leq m \leq 3).$$

## 5. Summary

We have examined in detail the cases  $N = 2$  and  $N = 3$  of exact solutions of the form

$$\psi(x) = \sum_{n=1}^N A_n x^{2n+\epsilon} \exp \left[ -\frac{1}{2}ax^2 - \frac{1}{4}bx^4 \right] \quad (\epsilon = 0, 1) \quad (5.1)$$

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<sup>2</sup>This follows from the fact that  $p < 0$ ,  $D = 64r - 16p^2 = -1024(a^4 + 12a^2b + 45b^2) < 0$  and the discriminant  $\Delta$  defined in [3, Eq. (1.11.17)] is  $\Delta = 3^2 \times 2^{16}(a^{12} + 36a^{10}b + 402a^8b^2 + 1848a^6b^3 + 3897a^4b^4 + 35100a^2b^5 + 40500b^6) > 0$ .

Table 2: Values of the coefficients when  $\lambda = 0.50$ ,  $\eta = 0.03$  and  $\omega^2 = 0.0625$  obtained from (4.3) and the corresponding odd eigenvalues  $E_{2m+1}$ .

$m$	$A_1^{(2m+1)}$	$A_2^{(2m+1)}$	$A_3^{(2m+1)}$	$E_{2m+1}$
0	+0.243487	+0.018657	+0.000453	1.144540
1	−0.611015	−0.100752	−0.003556	3.708044
2	−1.690968	+0.375069	+0.030907	6.947903
3	−2.941504	+1.800358	−0.271852	10.699513

of the Schrödinger equation (1.1), where  $a = \lambda(3/\eta)^{\frac{1}{2}}/4$  and  $b = (\eta/3)^{\frac{1}{2}}$  with  $\eta > 0$ . The cases  $N = 0$  and  $N = 1$  have been given earlier in [1] and [4], respectively. It is found that the solution (5.1) can only exist if a constraint on the coupling parameters  $\omega^2$ ,  $\lambda$  and  $\eta$  is satisfied, viz.

$$\gamma = 4N + 3 + 2\epsilon = \left(\frac{3}{\eta}\right)^{\frac{1}{2}} \left(\frac{3\lambda^2}{16\eta} - \omega^2\right).$$

For a given  $N$  and parity  $\epsilon$  of the eigenfunction, two parameters are free to be chosen with the third then fixed by the above constraint.

For each value of  $N$  considered it is found that  $N + 1$  eigenstates are produced with eigenvalues  $E_0, E_2, \dots, E_{2N}$  in the case of even modes and  $E_1, E_3, \dots, E_{2N+1}$  in the case of odd modes. We present below a summary of the ground-state eigenvalues and eigenfunctions of type (5.1) with  $\epsilon = 0$  and normalised such that  $A_0 = 1$ :

$$N = 0 : \quad E_0 = \frac{1}{2}a, \quad \gamma = 3,$$

$$N = 1 : \quad E_0 = \frac{3}{2}a - \sqrt{a^2 + 2b}, \quad \gamma = 7; \quad A_1 = \sqrt{a^2 + 2b} - a,$$

$$N = 2 : \quad E_0 = \frac{5}{2}a - P \sin(\theta + \frac{2}{3}\pi), \quad \gamma = 11; \quad A_1 = P \sin(\theta + \frac{2}{3}\pi) - 2a, \quad A_2 = \frac{2bA_1}{A_1 + 4a},$$

$$N = 3 : \quad E_0 = \frac{7}{2}a - \chi^*, \quad \gamma = 15,$$

where  $P$  and  $\theta$  are defined in (3.2) and  $\chi^*$  denotes the largest root of the quartic (4.2). In the case  $N = 4$  only numerical solutions for the coefficients  $A_n$  ( $1 \leq n \leq 3$ ) are obtained for a specific choice of parameters.

## References

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